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Multiobjectivization with NSGA-II on the Noiseless BBOB Testbed

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ABSTRACT

The idea of multiobjectivization is to reformulate a single-objective problem as a multiobjective one. In one of the scarce studies proposing this idea for problems in *continuous* domains, the distance to the closest neighbor (DCN) in the population of a multiobjective algorithm has been used as the additional (dynamic) second objective. As no comparison with other state-of-the-art single-objective optimizers has been presented for this idea, we have benchmarked two variants (with and without the second DCN objective) of the original NSGA-II algorithm using two different mutation operators on the noiseless BBOB’2013 testbed. It turns out that multiobjectivization helps for several of the 24 benchmark functions, but that, compared to the best algorithms from BBOB’2009, a significant performance loss is visible. Moreover, on some functions, the choice of the mutation operator has a stronger impact on the performance than whether multiobjectivization is employed or not.

Categories and Subject Descriptors

G.1.6 [Numerical Analysis]: Optimization—*global optimization, unconstrained optimization*; F.2.1 [Analysis of Algorithms and Problem Complexity]: Numerical Algorithms and Problems

General Terms

Algorithms, Experimentation

Keywords

Optimization; Benchmarking; Multiobjectivization

1. INTRODUCTION

The idea of multiobjectivization, i.e., the reformulation of a single-objective problem by multiple objectives and its resolution by means of a multiobjective optimizer, has been around since the beginning of the new millennium [14, 13].

Two basic ideas can thereby be distinguished: either the single-objective problem is *decomposed* into two or more objective functions [14] or one or more *additional* objective functions, so called *helper-objectives*, are optimized along with the original single-objective function [14, 13]. Several studies report on improving performance for combinatorial problems—early examples range from the traveling salesman problem [14], over reducing bloat in genetic programming [3, 6], to job-shop scheduling [13]. Also for some real-world optimization tasks, multiobjectivization seems to help [9]. The main argument in favor of multiobjectivization for combinatorial problems is thereby the ability to overcome local optima and the possibility of introducing additional search directions on plateaus of equal function values. This is related to the more general idea of increasing population diversity which has been studied independently, see e.g., [18].

Whereas the positive impact of multiobjectivization for combinatorial problems depends highly on the choice of typically problem-dependent objective functions [4, 10], for continuous problems, most studies favor a problem-independent approach, in which the diversity of the algorithm’s population or archive is used as a second objective function [2, 5, 16, 17]. The main argument of [16] to use the distance to the closest neighbor (DCN) in the population of the NSGA-II algorithm [7] as the second objective is that such an objective function “decreases the selection pressure of the original (single-objective) optimization scheme” with the result that “some low-quality individuals could survive in the population with a higher probability” and in turn “these individuals could help to avoid stagnation in local optima” [16].

Unfortunately, in [16, 17], no comparison with other state-of-the-art single-objective methods is performed. Here, we want not only to investigate the impact of multiobjectivization on the performance on the BBOB’2013 noiseless functions [8, 12], but also to see how the approach of [16] compares with state-of-the-art algorithms for numerical optimization. To this end, we used the original implementation of NSGA-II [1] with almost the same algorithmic components as described in [16]. More precisely, we used no additional termination criterion other than the maximum number of function evaluations, performed no restarts, and used the suggested uniform mutation operator as in [16]. In order to investigate the impact of multiobjectivization, we considered an NSGA-II variant with DCN as the second objective (U-DCN) and another where the second objective function was simply set to zero (U-zero). In order to have a better idea of how much the choice of mutation operators affects the search performance, we further compared with the variants

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where NSGA-II's original polynomial mutation [7] replaces the uniform mutation (denoted by P-DCN and P-zero).

More details on the algorithms are given in the next section while Sec. 3 details the experimental procedure. Section 4 presents the mandatory timing experiments and the comparison results in Sec. 5 conclude the paper.

2. ALGORITHM PRESENTATION

2.1 The Artificial Second Objective

There are several ways to introduce artificial objectives into a mono-objective problem, which are in general to be considered as functions measuring the diversity of a population of solutions. In [17], the authors studied the performance of three such functions, namely DCN (distance to the closest neighbor of the population), ADI (average distance to all individuals), and DBI (distance to the best individual). They showed that multiobjectivization with DCN as a second objective leads to superior performance.

In this study, we shall also use DCN for multiobjectivizing the BBOB functions. Having a set of individuals, the DCN with respect to individual i is defined as the Euclidean distance to the closest member of the population, considering that the decision space is real-valued. More formally,

$$\text{DCN}(i) = \min_{j \neq i} \left\{ \left(\sum_{\ell} (x_{\ell}^i - x_{\ell}^j)^2 \right)^{\frac{1}{2}} \right\},$$

where x_{ℓ}^i is the ℓ -th decision variable w.r.t. individual i .

2.2 The Multiobjective Algorithm

Among a multitude of multiobjective algorithms, we consider the well-known NSGA-II [7] which was also employed in [17]. For the sake of reproducibility, we recall the main components of NSGA-II and the way they were implemented in our experiments. First, since the standard NSGA-II deals with minimizing objectives while DCN is to be maximized (i.e. to increase diversity), we set the second objective used in NSGA-II to be the maximum DCN over all individuals minus the DCN of the considered individual. The population size N was set to be 8. In fact, a small population size was shown to perform relatively well in [17], and the population size in the standard NSGA-II implementation [1] should be a multiple of 4. We used the simulated binary crossover (SBX) with a distribution index of 15, where each gene (variable) was crossed with a probability of 0.5. As for the mutation, we considered two operators: (i) the uniform mutation (U), and (ii) the polynomial mutation (P) with distribution index $\eta = 100$. Notice that only the uniform mutation was considered in [17]. We set the crossover probability to 1 and the mutation probability to $1/D$, where D is the number of variables (i.e. problem dimension). While the uniform mutation naturally restricts the variables to an interval (here chosen as $[-5, 5]$), we also restricted the decision variables to this interval for the polynomial mutation by assigning all the mass of the probability distribution that is outside a variable's bound to the boundary value.

3. EXPERIMENTAL PROCEDURE

In order to study the impact of multiobjectivization, we considered running NSGA-II while artificially setting the second objective to zero, i.e., all individuals have equal values in their second objectives. This has the effect of turning

off the crowding-distance-based selection mechanism specific to NSGA-II, and favoring the selection of individuals having better fitness in the original first objective.

We ended up with four algorithm variants depending on whether DCN was switched on or off, and which mutation (U or P) was used. In the remainder, these variants are respectively denoted by U-DCN, U-zero, P-DCN, and P-zero.

For our experimentation, we used the standard C implementation of NSGA-II available for free download at [1], and set objectives and parameters to fit in our settings. Moreover, the initial population in NSGA-II was uniformly sampled in $[-5, 5]^D$. We run the four NSGA-II variants up to a budget of $10^6 D$ function evaluations or until the maximal BBOB precision of 10^{-8} was reached. It is to notice that there was no independent restart in our implementations. We considered dimensions $D \in \{2, 3, 5, 10, 20\}$ and all the 15 instances of the BBOB'2013 testbed.

4. TIMING EXPERIMENTS

In order to assess the dependency of the four algorithm variants on the problem dimension, the requested BBOB timing experiments were performed on a Dell XPS 720 machine using the Intel® Core™2 Quad Processor Q6600 running at 2.40 GHz with 2.0 GB RAM. Note that each implementation was deployed exclusively on a single core of the CPU. All implementations were built using the GCC 4.7.2 compiler and executed under the Ubuntu 12.10 Linux distribution. Each algorithm variant was run iteratively on the first instance of f_8 within $10^5 D$ function evaluations until *at least* 30 seconds had passed. This procedure was repeated over seven problem dimensions, i.e. $D = \{2, 3, 5, 10, 20, 40, 80\}$. The approximate per-function-evaluation runtimes for U-zero were 7.3, 7.8, 8.5, 10, 13, 18, and 29 times 10^{-7} seconds; for U-DCN 12, 13, 16, 23, 38, 90, and 110 times 10^{-7} seconds; for P-zero 9.9, 10, 11, 13, 16, 23, and 36 times 10^{-7} seconds; and for P-DCN 13, 16, 19, 28, 42, 58, and 82 times 10^{-7} seconds in 2, 3, 5, 10, 20, 40, and 80 dimensions respectively.

5. RESULTS

Results from experiments according to [11] on the benchmark functions given in [8, 12] are presented in Figures 1, 2, and 3 and in Tables 1 and 2. The **expected running time (ERT)** used in the figures and tables depends on a given target function value, $f_t = f_{\text{opt}} + \Delta f$, and is computed over all relevant trials as the number of function evaluations executed during each trial while the best function value did not reach f_t , summed over all trials and divided by the number of trials that actually reached f_t [11, 15]. **Statistical significance** is tested with the rank-sum test for a given target Δf_t (10^{-8} as in Figure 1) using, for each trial, either the number of needed function evaluations to reach Δf_t (inverted and multiplied by -1), or, if the target was not reached, the best Δf -value achieved, measured only up to the smallest number of overall function evaluations for any unsuccessful trial under consideration.

From our experimental results, three main observations are formulated in the following paragraphs. All results mentioned are statistically significant. When comparing two algorithm variants, the significance is thereby checked with the two-algorithm facilities of BBOB (plots not shown here).

Impact of Multiobjectivization: When comparing the DCN variants to the variants without DCN, multiobjectiviza-

tion seems to help on some functions whereas a negative impact can only be observed for a few. For NSGA-II with uniform mutation, U-DCN outperforms U-zero for the separable functions, original Rosenbrock (f_8), and the discus function (f_{11} in 20-D) for almost all targets and on f_{14} and f_{22} in 5-D and for the most difficult targets. U-zero is, on the other hand, only better on f_{14} and the separable functions (in 20-D) for easy targets. For NSGA-II with polynomial mutation, the impact of multiobjectivization is less pronounced with a similar tendency on the separable and moderate functions with P-DCN being better only on f_2, f_6, f_8 , and f_{14} . On the sphere (f_1) and the linear function f_5 , on the other hand, the version without the DCN objective is clearly better. The performance differences are larger in higher dimensions.

Impact of Mutation: On some functions, the choice of mutation operator seems to have a stronger impact on the performance than whether multiobjectivization is employed or not. Specifically, when comparing U-DCN with P-DCN, the polynomial mutation gives better results on functions $f_1, f_2, f_5, f_6, f_{11}$, and f_{14} for the most difficult targets while the uniform mutation is typically better at the beginning of the search. This behavior is not surprising as the distribution index $\eta = 100$ for the polynomial mutation was fixed throughout the search. As a result, the mutation's step size is typically too small at the beginning of the search but better suited at later stages. On f_{20}, f_{21} , and f_{22} , the uniform mutation is, however, interestingly better for all targets.

Competitiveness: The third observation is that the ERTs of all the four variants are still far from being competitive with the artificial best algorithm from BBOB'2009.

It is worth noticing that, initially, our goal was *not* to design an optimizer that would perform competitively compared to existing state-of-the-art single-objective algorithms, which normally use advanced optimization techniques like step size adaptation. Notice also that we have conducted some other preliminary experiments using another advanced multiobjective algorithm, namely R2-EMOA, and we observed better ERTs, but a seemingly comparable impact of DCN. This suggests that DCN or any other alternative objectives may, to some extent, be beneficial if carefully combined with an appropriate multiobjective algorithm or variation operator (e.g. a specific mutation). From our experiments, however, we can only conclude that using DCN with the specified NSGA-II is showing a limited potential for tackling the noiseless single-objective BBOB testbed.

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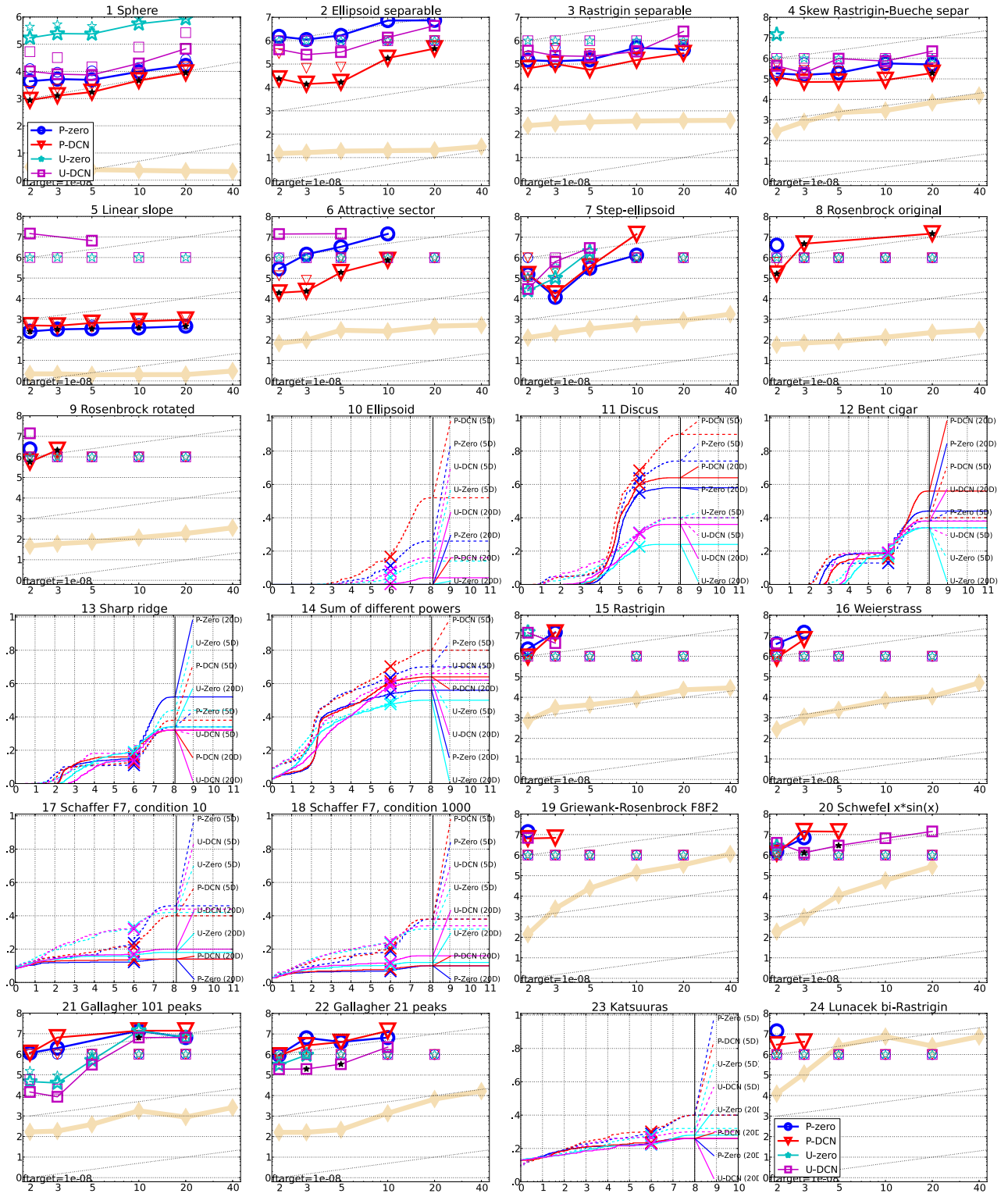


Figure 1: Expected running time (ERT in number of f -evaluations) divided by dimension for target function value 10^{-8} as \log_{10} values versus dimension. Different symbols correspond to different algorithms given in the legend of f_1 and f_{24} . Light symbols give the maximum number of function evaluations from the longest trial divided by dimension. Horizontal lines give linear scaling, slanted dotted lines give quadratic scaling. Black stars indicate statistically better results compared to all other algorithms with $p < 0.01$ and Bonferroni correction number of dimensions (six). For f_{10} – f_{13} , f_{17} , f_{18} , and f_{23} , no finite ERT could be displayed (except for 2-D), therefore the empirical cumulative distribution graphs per function (similar to Figs. 2 and 3) are shown instead (5-D: dashed lines; 20-D: straight lines). Legend: \circ :P-zero, ∇ :P-DCN, \star :U-zero, \square :U-DCN

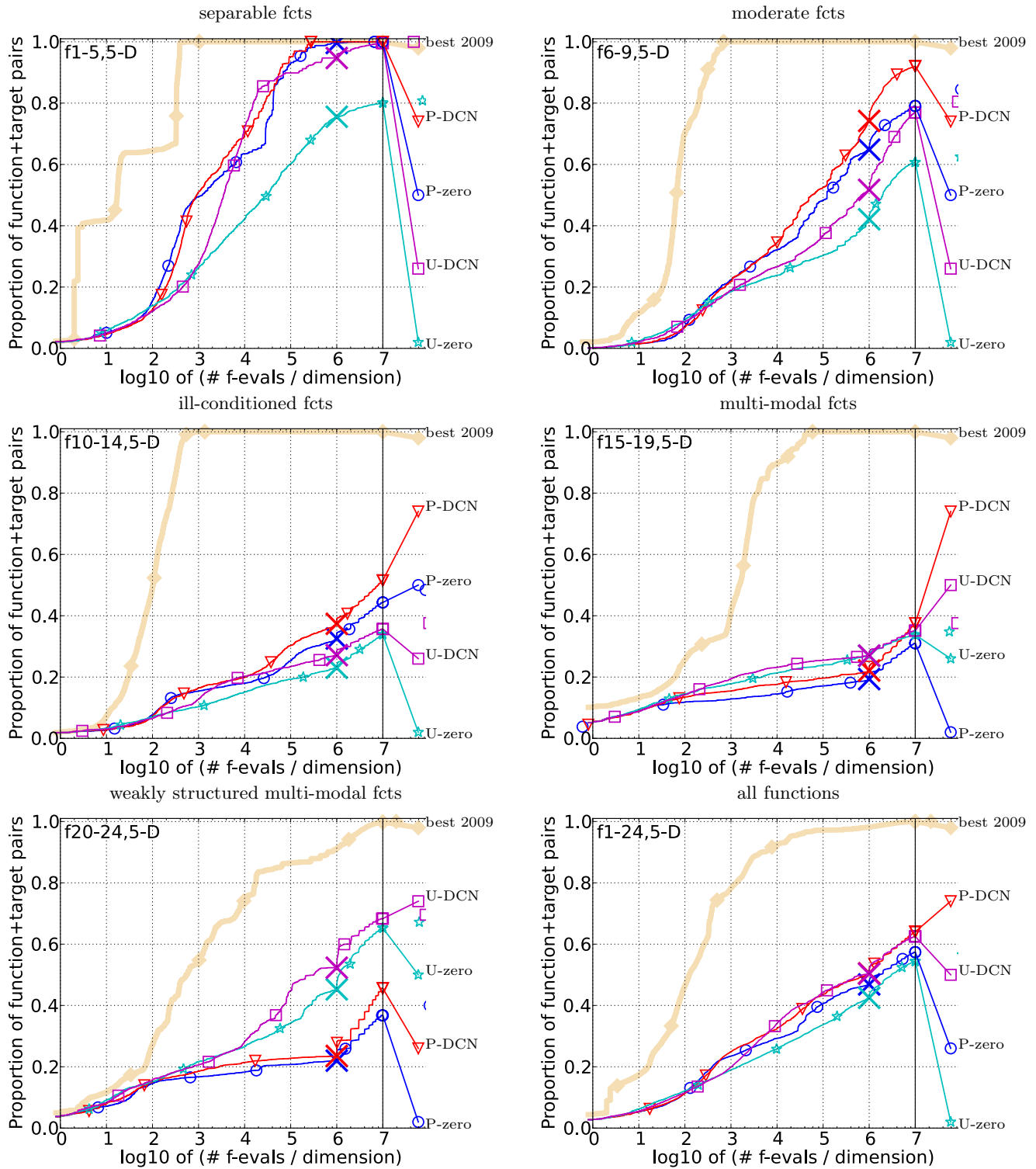


Figure 2: Bootstrapped empirical cumulative distribution of the number of objective function evaluations divided by dimension (FEvals/D) for 50 targets in $10^{[-8..2]}$ for all functions and subgroups in 5-D. The “best 2009” line corresponds to the best ERT observed during BBOB 2009 for each single target.

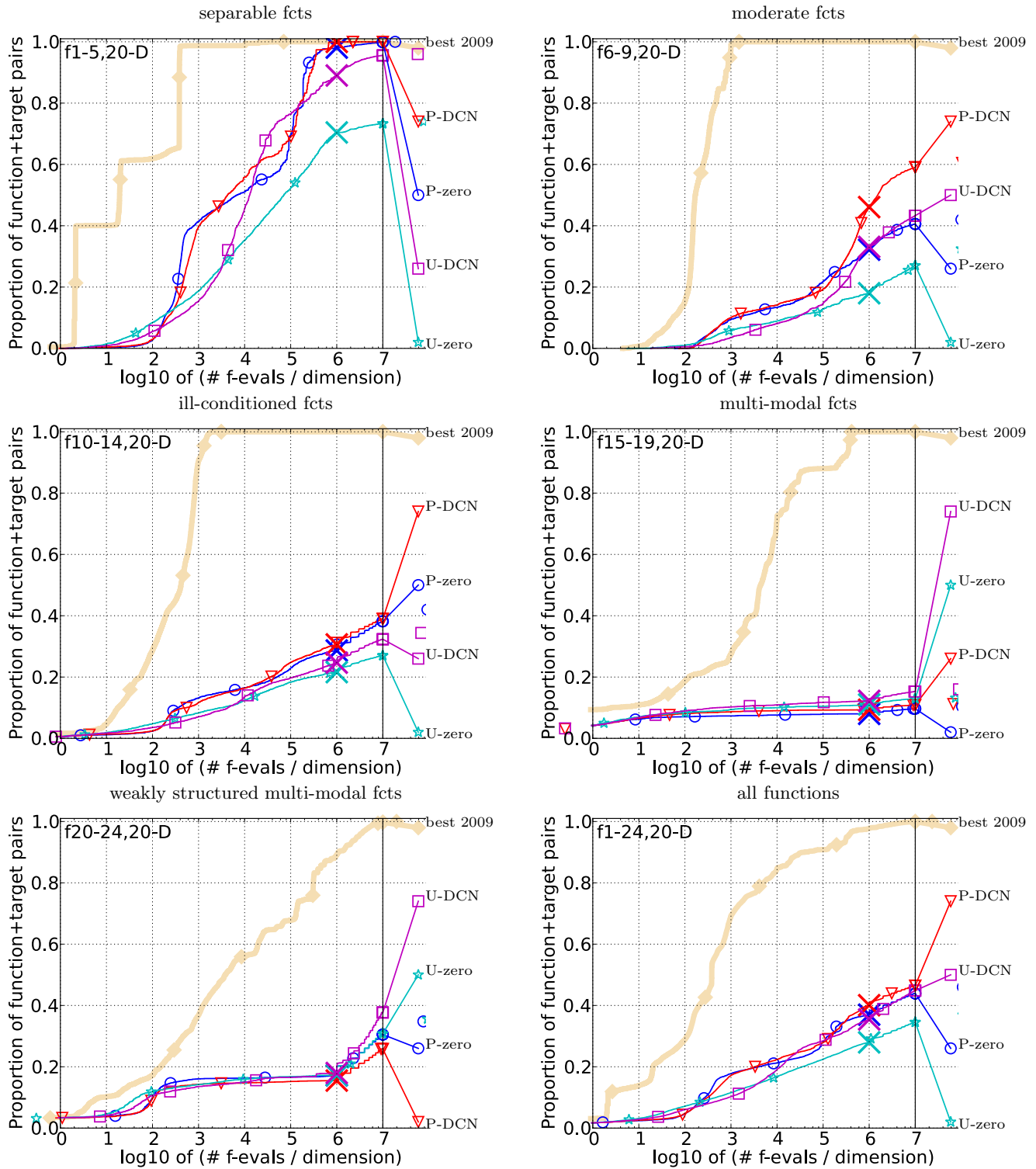


Figure 3: Bootstrapped empirical cumulative distribution of the number of objective function evaluations divided by dimension (FEvals/D) for 50 targets in $10^{[-8..2]}$ for all functions and subgroups in 20-D. The “best 2009” line corresponds to the best ERT observed during BBOB 2009 for each single target.

Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f1	11	12	12	12	12	12	15/15	f13	132	195	250	1310	1752	2255	15/15
P-0	11(14)	29(26)	41(30)	55(27)	133(35)* ²	664(236)	15/15	P-0	7.6e4(9e4)	1.7e5(2e5)	2.8e5(3e5)	∞	∞	∞ 5e6	0/15
P-D	7.5(12)	28(20)	41(20)	64(25)	199(48)	494(147)	15/15	P-D	3.4e4(4e4)	3.6e5(4e5)	2.8e5 (3e5)	∞	∞	∞ 5e6	0/15
U-0	3.2(3)	14(8)	54(43)	381(303)	4039(2899)	3.9e4(2e4)	15/15	U-0	2.5e4(4e4)	1.7e5(2e5)	2.8e5(3e5)	∞	∞	∞ 5e6	0/15
U-D	3.1 (3)	21(11)	84(45)	367(148)	848(548)	2283(1702)	15/15	U-D	1.4e4 (2e4)	5.1e4 (6e4)	2.8e5(3e5)	∞	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f2	83	87	88	90	92	94	15/15	f14	10	41	58	139	251	476	15/15
P-0	16(10)	21(11)	44(32)	225(135)	2835(3668)	1.8e4(2e4)	7/15	P-0	3.1(4)	11(7)	11(5)	80(54)	∞	∞ 5e6	0/15
P-D	20(9)	27(9)	47(22)	139(99)	212 (111)* ³	610 (818)*	15/15	P-D	2.0(2)	9.3(4)	10 (3)	113(191)	5.4e4 (5e4)	∞ 5e6	0/15
U-0	175(246)	780(964)	1961(1617)	2.0e4(2e4)	∞	∞ 5e6	0/15	U-0	1.5(2)	3.6 (2)	11(6)	2.6e5(3e5)	∞	∞ 5e6	0/15
U-D	107(160)	193(191)	264(225)	1229(858)	1345(776)	7404(6761)	12/15	U-D	1.2 (0.7)	6.8(5)	23(12)	1269(1067)	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f3	716	1622	1637	1646	1650	1654	15/15	f15	511	9310	19369	20073	20769	21359	14/15
P-0	9.4(9)	44(27)	111(41)	111(41)	116(41)	223(75)	15/15	P-0	4.5e4(5e4)	∞	∞	∞	∞	∞ 5e6	0/15
P-D	7.5(11)	26(29)	152(178)	152(177)	155(176)	166 (173)	15/15	P-D	6.4e4(8e4)	∞	∞	∞	∞	∞ 5e6	0/15
U-0	0.62 (0.5)	2.4 (1)	8.4(5)	88(61)	1027(650)	∞ 5e6	0/15	U-0	1553(2758)	∞	∞	∞	∞	∞ 5e6	0/15
U-D	0.85(0.5)	3.6(2)	5.4 (3)	10 (4)* ²	65 (62)	374(678)	13/15	U-D	516 (62)	2163 (2685)	∞	∞	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f4	809	1633	1688	1817	1886	1903	15/15	f16	120	612	2662	10449	11644	12095	15/15
P-0	24(9)	81(40)	139(52)	131(49)	132(58)	256(95)	15/15	P-0	5.5(16)	7997(1e4)	1.2e4(2e4)	∞	∞	∞ 5e6	0/15
P-D	42(46)	74(116)	210(286)	195(266)	189(258)	189 (255)	15/15	P-D	1.1 (0.7)	1846(4086)	7522(8462)	3290 (3652)	∞	∞ 5e6	0/15
U-0	0.76 (0.4)	4.7 (2)	12(10)	112(97)	1134(559)	∞ 5e6	0/15	U-0	1.3(0.9)	600(1235)	2030(2812)	∞	∞	∞ 5e6	0/15
U-D	1.1(0.7)	6.1(4)	8.2 (6)	12 (6)* ²	89 (192)	876(1352)	8/15	U-D	1.7(1.0)	90 (82)	1913 (2389)	∞	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f5	10	10	10	10	10	10	15/15	f17	5.2	215	899	3669	6351	7934	15/15
P-0	50(58)	101 (85)	113 (84)	129 (84)	146 (81)	162 (81)* ³	15/15	P-0	5.6(8)	3.3e4(4e4)	2.3e4(3e4)	∞	∞	∞ 5e6	0/15
P-D	37(29)	118(77)	132(74)	168(69)	219(50)	290(56)	15/15	P-D	4.3(4)	4.7e4(6e4)	7.8e4(9e4)	∞	∞	∞ 5e6	0/15
U-0	19 (15)	217(97)	2697(1161)	2.5e5(1e5)	∞ 5e6	∞ 5e6	0/15	U-0	2.6 (2)	33(56)	4877 (6051)	∞	∞	∞ 5e6	0/15
U-D	25(18)	240(136)	988(587)	3.7e4(8e4)	2.6e5(4e5)	1.5e6(2e6)	2/15	U-D	4.1(4)	6.0 (9)	5087(8339)	∞	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f6	114	214	281	580	1038	1332	15/15	f18	103	378	3968	9280	10905	12469	15/15
P-0	7.3(7)	6.4 (5)	8.8(10)	109(104)	1887(2491)	6888(8649)	4/15	P-0	933(2547)	4.0e4(5e4)	1.8e4(2e4)	∞	∞	∞ 5e6	0/15
P-D	7.6(7)	7.6(4)	8.3 (5)	23 (16)	75 (88)	410 (303)* ³	14/15	P-D	193(272)	2.9e4(3e4)	8439(9400)	∞	∞	∞ 5e6	0/15
U-0	6.5 (7)	77(174)	1678(1399)	8123(9346)	9430(1e4)	∞ 5e6	0/15	U-0	2.1 (2)	7032(1e4)	8805(1e4)	∞	∞	∞ 5e6	0/15
U-D	7.7(9)	96(189)	293(492)	2097(4324)	3875(5032)	5.3e4(6e4)	1/15	U-D	2.5(3)	6288 (7517)	8405 (1e4)	∞	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f7	24	324	1171	1572	1597	1597	15/15	f19	1	1	242	1.2e5	1.2e5	1.2e5	15/15
P-0	217(261)	2798(7738)	1284 (2159)	983 (1602)	983 (1646)	968 (1595)	13/15	P-0	27(16)	2.2e6(3e6)	3.1e5(3e5)	∞	∞	∞ 5e6	0/15
P-D	113(233)	1695(2334)	1569(2303)	1175(1774)	1175(1813)	1158(1747)	12/15	P-D	44(40)	5.0e5(9e5)	6.5e4(7e4) 619 (706)	618 (640)	∞ 5e6	0/15	
U-0	12 (13)	1488(2964)	2969(3523)	6279(6881)	6279(7379)	6188(6416)	6/15	U-0	27 (20)	7251 (6858)	3.2e4 (4e4) 3	∞	∞ 5e6	0/15	
U-D	20(22)	1132 (41)	1421(2326)	9508(1e4)	9508(1e4)	9362(1e4)	4/15	U-D	32(30)	1.3e4(1e4)	4.9e4(5e4)	∞	∞ 5e6	0/15	
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f8	73	273	336	391	410	422	15/15	f20	16	851	38111	54470	54861	55313	14/15
P-0	18(19)	2863(9152)	3230 (7738)	1.8e5(2e5)	1.7e5(2e5) ∞ 5e6	∞ 5e6	0/15	P-0	15(10)	371(909)	∞	∞	∞	∞ 5e6	0/15
P-D	23(24)	6797(9224)	6645(7523)	9250 (6704)	1.4e4 (6794) ∞ 5e6	∞ 5e6	0/15	P-D	8.6(8)	336(965)	1837(2198)	1285(1423)	1276(1504)	1266(1559)	1/15
U-0	16 (9)	7820(1e4)	∞	∞	∞	∞ 5e6	0/15	U-0	4.0 (2)	17(29)	527(655)	369(413)	371(413)	390(411)	0/15
U-D	17(16)	713 (1962)	4007(4023)	1.9e5(2e5)	∞ 5e6	∞ 5e6	0/15	U-D	4.5(3)	5.6 (8)	367 (459)	257 (321)	255 (276)	261 (332)	4/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f9	35	127	214	300	335	369	15/15	f21	41	1157	1674	1705	1729	1757	14/15
P-0	685(1900)	3145 (607)	3476 (1371)	2.4e5(3e5) ∞	2.4e5(3e5) ∞ 5e6	2.0e5(2e5) ∞ 5e6	0/15	P-0	8713(3)	1.2e4(2e4)	∞	∞	∞	∞ 5e6	0/15
P-D	446(980)	2.7e4(4e4)	1.7e4(2e4)	1.9e4 (2e4) 1.1e4 (5e4) 1.2e5 (2e5)	∞ 5e6	∞ 5e6	0/15	P-D	1.9e4(6e4)	1.2e4(2e4)	∞	∞	∞	∞ 5e6	0/15
U-0	378(55)	1.4e4(2e4)	3.4e5(4e5)	∞	∞ 5e6	∞ 5e6	0/15	U-0	2.9(1)	236 (396)	1409(2021)	1386(1897)	1376(1871)	1398(1834)	11/15
U-D	65 (59)	3.1e4(4e4)	2.3e4(4e4)	7.2e4(8e4) ∞	∞ 5e6	∞ 5e6	0/15	U-D	2.7 (3)	435(447)	906 (1582)	892 (1553)	885 (1532)	880 (1503)	12/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f10	349	500	574	626	829	880	15/15	f22	71	386	938	1008	1040	1068	14/15
P-0	2.3e4(2e4)	1.4e5(2e5)	∞	∞	∞	∞ 5e6	0/15	P-0	2.6e4(4e4) 3.6e4 (5e4)	2.1e4(3e4)	2.0e4(3e4)	1.9e4(2e4)	1.9e4(2e4)	1.9e4(2e4)	3/15
P-D	7421 (8952)	6.6e4 (8e4)	1.3e5 (1e5)	1.2e5 (1e5)	∞	∞ 5e6	0/15	P-D	2.6e4(4e4) 9e4 (3e4)	2.1e4(2e4)	2.0e4(2e4)	1.9e4(2e4)	1.9e4(2e4)	1.9e4(2e4)	3/15
U-0	2.1e5(2e5)	∞	∞	∞	∞	∞ 5e6	0/15	U-0	4.8(6)	1106(458)	1458(2755)	1832(2640)	5409(5342)	3.5e4(4e4)	0/15
U-D	9.9e4(1e5)	∞	∞	∞	∞	∞ 5e6	0/15	U-D	1.9 (1)	327 (164)	1146 (2667)	1129 (2513)	1256 (2478)	1469 (2343)	13/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f11	143	202	763	1177	1467	1673	15/15	f23	3.0	518	14249	31654	33030	34256	15/15
P-0	781(856)	1264(814)	479(270)	797(227)	5.1e4(5e4) ∞ 5e6	∞ 5e6	0/15	P-0	3.6(3)	29(62)	2398(2711)	∞	∞	∞ 5e6	0/15
P-D	606(502)	896 (629)	312 (165)	619 (310)	1.2e4 (1e4) ∞ 5e6	∞ 5e6	0/15	P-D	3.6(3)	3.2 (4)	970 (1228)	∞	∞	∞ 5e6	0/15
U-0	447(538)	2846(3800)	9233(8613)	∞	∞	∞ 5e6	0/15	U-0	3.2 (2)	43(50)	1527(1678)	∞	∞	∞ 5e6	0/15
U-D	239 (214)	2586(3033)	8587(9687)	∞	∞	∞ 5e6	0/15	U-D	3.4(2)	10(17)	∞	∞	∞	∞ 5e6	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f12	108	268	371	461	1303	1494	15/15	f24							

Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f1	43	43	43	43	43	43	15/15	f13	652	2021	2751	18749	24455	30201	15/15
P-0	45(15)	63(15)	71 (12)	100 (22)* ³	337 (79)* ⁴	2140 (550)	15/15	P-0	2.7e4(5e4)	4.0e4(5e4)	4.7e4 (6e4)	1.6e4 (2e4)	∞	∞ 2e7	0/15
P-D	44(14)	63(16)	74(17)	147(38)	702(169)	2421(596)	15/15	P-D	1.5e4(3e4)	1.5e4 (2e4)	1.0e5(1e5)	∞	∞	∞ 2e7	0/15
U-0	8.6 (2)* ⁴	37 (10)* ³	119(26)	1029(335)	1.0e4(3953)	1.1e5(3e4)	13/15	U-0	5052 (2e4)	1.6e4(2e4)	1.0e5(1e5)	∞	∞	∞ 2e7	0/15
U-D	21(5)	111(30)	378(97)	1589(413)	4632(1517)	1.0e4(2945)	15/15	U-D	2.7e4(3e4)	4.0e4(5e4)	1.0e5(1e5)	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f2	385	386	387	390	391	393	15/15	f14	75	239	304	932	1648	15661	15/15
P-0	22 (3)	28 (10)	69(64)	637(212)	4598(2229)	1.6e5(2e5)	2/15	P-0	25(9)	15(4)	14(3)	3203(2945)	∞	∞ 2e7	0/15
P-D	25(6)	32(6)	52 (14)	257 (164)* ³	1017 (1191)* ²	8815 (4266)* ²	2/15	P-D	22(8)	15(3)	14 (2)	1149 (358)*	∞	∞ 2e7	0/15
U-0	305(203)	1104(917)	4015(1999)	2.8e4(1e4)	∞	∞ 2e7	0/15	U-0	4.0 (2)*	6.2 (2)* ⁴	22(7)	∞	∞	∞ 2e7	0/15
U-D	130(48)	374(352)	563(333)	1535(1733)	2.1e4(3e4)	9.5e4(1e5)	3/15	U-D	6.6(3)	15(3)	54(15)	2814(1041)	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f3	5066	7626	7635	7643	7646	7651	15/15	f15	30378	1.5e5	3.1e5	3.2e5	4.5e5	4.6e5	15/15
P-0	216(157)	283(156)	378(171)	378(171)	381(169)	508 (147)	15/15	P-0	∞	∞	∞	∞	∞	∞ 2e7	0/15
P-D	145(75)	331(590)	753(690)	752(689)	752(689)	751(688)	15/15	P-D	∞	∞	∞	∞	∞	∞ 2e7	0/15
U-0	1.4 (0.3)* ³	4.8 (2)* ²	15(6)	170(81)	1900(536)	∞ 2e7	0/15	U-0	∞	∞	∞	∞	∞	∞ 2e7	0/15
U-D	2.7(1)	9.5(5)	13 (4)	43 (26)* ⁴	138 (117)* ²	2113(2614)	5/15	U-D	∞	∞	∞	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f4	4722	7628	7666	7700	7758	1.4e5	9/15	f16	1384	27265	77015	1.9e5	2.0e5	2.2e5	15/15
P-0	271(152)	263(100)	375(134)	373(134)	374(130)	28(7)	15/15	P-0	2.3e4(3e4)	∞	∞	∞	∞	∞ 2e7	0/15
P-D	214(249)	337(323)	447(313)	445(312)	444(309)	25 (17)	15/15	P-D	2687 (7228)	∞	∞	∞	∞	∞ 2e7	0/15
U-0	2.1 (0.6)* ²	6.9 (2)* ²	25(6)	255(69)	3201(1837)	∞ 2e7	0/15	U-0	7416(1e4)	∞	∞	∞	∞	∞ 2e7	0/15
U-D	3.9(2)	12(8)	16(6)* ²	42 (15)* ³	219 (173)	120(119)	5/15	U-D	2759(7235)	∞	∞	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f5	41	41	41	41	41	41	15/15	f17	63	1030	4005	30677	56288	80472	15/15
P-0	128(20)	163 (32)	174 (31)	185 (30)* ³	200 (27)* ⁴	217 (29)* ⁴	15/15	P-0	2.3e4(7)	∞	∞	∞	∞	∞ 2e7	0/15
P-D	141(31)	190(46)	204(47)	248(29)	336(37)	423(35)	15/15	P-D	2.6(2)	∞	∞	∞	∞	∞ 2e7	0/15
U-0	92 (24)* ²	858(334)	1.1e4(2402)	∞	∞	∞ 2e7	0/15	U-0	1.7 (0.7)	∞	∞	∞	∞	∞ 2e7	0/15
U-D	174(77)	942(290)	3043(1126)	1.6e4(5662)	1.3e6(1e6)	∞ 2e7	0/15	U-D	1.9(1)	∞	∞	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f6	1296	2343	3413	5220	6728	8409	15/15	f18	621	3972	19561	67569	1.3e5	1.5e5	15/15
P-0	9.2 (4)	826(2658)	3139(4113)	1.6e4(2e4)	∞ 2e7	∞ 2e7	15/15	P-0	∞	∞	∞	∞	∞	∞ 2e7	0/15
P-D	16(3)	17 (35)	471 (100)	505 (2293)	4356 (4185)	1.1e4 (1e4)	10/15	P-D	7.5e4(8e4)	∞	∞	∞	∞	∞ 2e7	0/15
U-0	3013(7731)	7233(1e4)	8.2e4(1e5)	5.5e4(6e4)	∞ 2e7	∞ 2e7	0/15	U-0	1.7e4 (2e4)	∞	∞	∞	∞	∞ 2e7	0/15
U-D	358(384)	2573(3520)	3426(3787)	6585(6094)	∞	∞ 2e7	0/15	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f7	1351	4274	9503	16524	16524	16969	15/15	f19	1	1	3.4e5	6.2e6	6.7e6	6.7e6	15/15
P-0	2528 (2942)	1.7e4 (2e4)	∞	∞	∞	∞ 2e7	15/15	P-0	3.2e7(4e7)	∞	∞	∞	∞	∞ 2e7	0/15
P-D	1.7e4(2e4)	∞	∞	∞	∞	∞ 2e7	0/15	P-D	4.3e6(1e7)	∞	∞	∞	∞	∞ 2e7	0/15
U-0	2.1e4(2e4)	∞	∞	∞	∞	∞ 2e7	0/15	U-0	317 (200)	∞	∞	∞	∞	∞ 2e7	0/15
U-D	1.8e4(2e4)	∞	∞	∞	∞	∞ 2e7	0/15	U-D	397(212)	2.9e8 (3e8)	∞	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f8	2039	3871	4040	4219	4371	4484	15/15	f20	82	46150	3.1e6	5.5e6	5.6e6	5.6e6	14/15
P-0	215 (128)	653 (65)	959 (81)	∞	4371	4484	15/15	P-0	22(5)	77(145)	∞	∞	∞	∞ 2e7	0/15
P-D	563(316)	1531(2658)	1717(2556)	2408 (2397)	3376 (2378)*	1.3e4 (1e4)	0/15	P-D	22(5)	11(14)	∞	∞	∞	∞ 2e7	0/15
U-0	1382(1246)	2430(2967)	2.0e4(2e4)	∞	∞	∞ 2e7	0/15	U-0	9.4 (4)	0.32 (0.3)	∞	∞	∞	∞ 2e7	0/15
U-D	551(579)	767(712)	1198(748)	2494(967)	3.3e4(4e4)	∞ 2e7	0/15	U-D	16(6)	1.4(0.6)	93 (103)	52 (56)	51 (56)	51 (53)	1/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f9	1716	3102	3277	3455	3594	3727	15/15	f21	561	6541	14103	14643	15567	17589	15/15
P-0	412 (296)* ²	4135 (3363)	6472 (6053)	∞	∞	∞ 2e7	15/15	P-0	5484 (2e4)	2.0e4(2e4)	9218 (1e4)	8878 (1e4)	8351 (9636)	7392 (9096)	2/15
P-D	1600(1101)	5404(3702)	∞	∞	∞	∞ 2e7	0/15	P-D	2.4e4(4e4)	4.3e4(5e4)	0e4(2e4)	1.9e4(2e4)	1.8e4(2e4)	1.6e4(2e4)	1/15
U-0	4.0e4(4e4)	∞	∞	∞	∞	∞ 2e7	0/15	U-0	1.8e4(4e4)	2.0e4 (2e4)	9218 (1e4)	8880(1e4)	8356(1e4)	7407(9096)	2/15
U-D	1.7e4(1e4)	9.3e4(1e5)	∞	∞	∞	∞ 2e7	0/15	U-D	8910(2e4)	2.0e4(2e4)	9219 (1e4)	8882(1e4)	8362(1e4)	7414(9094)	2/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f10	7413	8661	10735	14920	17073	17476	15/15	f22	467	5580	23491	24948	26847	1.3e5	12/15
P-0	∞	∞	∞	∞	∞	∞ 2e7	15/15	P-0	2.9e4(4e4)	2.3e4(3e4)	∞	∞	∞	∞ 2e7	0/15
P-D	∞	∞	∞	∞	∞	∞ 2e7	0/15	P-D	3.7e4(4e4)	2.3e4 (3e4)	∞	∞	∞	∞ 2e7	0/15
U-0	∞	∞	∞	∞	∞	∞ 2e7	0/15	U-0	2.1e4 (4e4)	∞	∞	∞	∞	∞ 2e7	0/15
U-D	∞	∞	∞	∞	∞	∞ 2e7	0/15	U-D	2.9e4(4e4)	∞	∞	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f11	1002	2228	6278	9762	12285	14831	15/15	f23	3.2	1614	67457	4.9e5	8.1e5	8.4e5	15/15
P-0	676(222)	617(254)	349(109)	1233(468)	∞	∞ 2e7	15/15	P-0	2.1(2)	2304(5644)	∞	∞	∞	∞ 2e7	0/15
P-D	457 (265)	417 (172)	211 (78)* ²	850 (385)	∞	∞ 2e7	0/15	P-D	2.1 (2)	1083 (876)	∞	∞	∞	∞ 2e7	0/15
U-0	1213(560)	7556(4994)	∞	∞	∞	∞ 2e7	0/15	U-0	2.4(3)	3936(6286)	∞	∞	∞	∞ 2e7	0/15
U-D	786(641)	3035(2507)	6177(5055)	∞	∞	∞ 2e7	0/15	U-D	2.1(2)	4711(6313)	∞	∞	∞	∞ 2e7	0/15
Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ	Δf_{opt}	1e1	1e0	1e-1	1e-3	1e-5	1e-7	#succ
f12	1042	1938	2740	4140	12407	13827	15/15	f24	1.3e6	7.5e6	5.2e7	5.2e7	5.2e7	5.2e7	3/15
P-0	699														